

Stability and Renormalization of Yang-Mills theory with Background Field Method: a Regularization Independent Proof

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Abstract

In this paper the stability and the renormalizability of Yang-Mills theory in the Background Field Gauge are studied. By means of Ward Identities of Background gauge invariance and Slavnov-Taylor Identities the stability of the classical model is proved and, in a regularization independent way, its renormalizability is verified. A prescription on how to build the counterterms is given and the possible anomalies which may appear for Ward Identities and for Slavnov-Taylor Identities are shown.

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1 Introduction

The Standard Model (SM) of elementary particles, based on non-abelian gauge theories, is known to be the most reliable model because of its predictive power. This is mainly due to the possibility of performing calculations in the perturbative regime. In turns this possibility is intimately linked to its renormalizability. In spite of these well consolidated results the calculations of higher order corrections are far from being practical because of the large amount of Feynman rules and of the loss of explicit gauge invariance in the quantization of the model.

Although the classical theory is characterized by the explicit gauge invariance, one has to chose a definite gauge to perform the quantization and the unavoidable renormalization needed to obtain meaningful results. By means of BRS symmetry [7], which has to be set in place of gauge invariance at the quantum level, one is able to show that the physical observables as S-matrix, form factors, masses etc. are gauge independent in the conventional formalism, but, nevertheless, the elementary building blocks of the quantum theory, *i.e.* one particle irreducible Green functions, are gauge dependent and gauge non-invariant. We remind the reader about the distinction between the gauge independence, *i.e.* the independence of relevant quantities from the gauge fixing parameters, and the gauge invariance of Green functions under gauge transformations expressed by Ward Identities.

Thus, there exists no evidence, throughout all intermediate steps of perturbative calculations of physical quantities, that every contribution has been properly taken into account, nor there is any simple test to check the correctness at each passage.

The computation of Green functions for elettroweak processes is a typical situation in which the use of gauge-invariant effective action would be advantageous to reduce the complexity of the Standard Model. We wish to underline that we mean here the gauge invariance valid before spontaneous symmetry breaking or gauge invariance under non homogeneous linear transformations for scalar fields.

To solve the problem of the explicit gauge-invariance breakdown upon quantization, the Background Field Method (BFM) was developed. By means of the BFM one can fix the gauge for quantum fields while keeping the gauge invariance of the effective action. In this way the Green functions derived from this new effective action are manifestly gauge invariant. Moreover, to build the S-matrix elements, one can choose a gauge fixing for background fields which is completely independent from that used for quantization and is more suitable to decouple the unphysical degree of freedom. The equivalence between physical quantities calculated in the conventional and in the BFM approach is proved in [10] and recently reviewed by C.Becchi on a rigorous ground.

This method, largely employed in quantum gravity and supergravity calculations [9], in gauge theories [10] and in supersymmetric models [13], has recently been applied with very encouraging results to the Standard Model [14], showing how to build gauge invariant Green functions along all intermediate steps of the calculations.

In spite of the large amount of renewed interest in the BFM, a regularization independent proof of renormalizability of gauge theories quantized with Background Gauge Fixing

is still lacking. As a consequence, there exists no test of compatibility among the various regularization schemes allowed with BFM on different models (see [14] and [12]) nor a verification of the absence of new anomalies.

This paper is devoted to a regularization independent analysis of the BFM approach of $SU(N)$ pure Yang-Mills models. The reader might object that in this situation the analysis is trivial thanks to Dimensional Regularization compatible with every symmetries of the model, but we have to meet to this objection that our analysis is directly extensible to chiral Yang-Mills models and, up to some little changes, to the SM. In these latter situations the help of an invariant regularization fails and one is entitled to fear that the introduction of the Background fields could generate new anomalies independent from which are associated with Quantum Gauge Fields. Therefore our study turns to be necessary. It supplies a complete answer to the anomaly problem for chiral Yang Mills model and lets to overcome some difficulties of SM (the study of a non-semi-simple gauge model in a spontaneously broken phase is in preparation).

In the first part we study the model in the tree level approximation. We want to verify two important features of BFM: the stability with respect to the splitting of the Gauge Field into the Background Field V_μ^a and Quantum Field A_μ^a and the non-renormalization properties of Background Field V_μ^a (the latter is the usual result [10] that the wave function renormalization $(V_0)_\mu^a = (Z_V)^{-\frac{1}{2}} V_\mu^a$ of this field coincides with the gauge coupling renormalization $g_0 = Z_g g$:

$$(Z_V)^{-\frac{1}{2}} = Z_g$$

as in QED). We quantize adopting the Background Gauge Fixing by means of BRS symmetry technique and distinguish at this level the different role of Quantum Gauge Field and Background Field: the BRS transformation of the former is the usual covariant derivative of $\Phi\Pi$ ghost while the BRS transformation of the latter, that is naively expected to vanish, yields a new BRS invariant ghost for the model. As it stands, this construction is completely fixed by BRS and Background Gauge Invariance (and by auxiliary field equations for ghosts and Lagrange multiplier) and the tree level model is parametrized by arbitrary constants which can be written in terms of the conventional three renormalization constants fixed by the normalization conditions.

In order to provide a regularization-independent proof of renormalizability, we work in the BPHZL ([6],[5]) framework. Avoiding any specific regularization we are allowed to spoil some symmetries of the model and, according to the Quantum Action Principle (QAP) ([2]), all sorts of breaking could affect the Slavnov-Taylor (ST) identities, the Ward Identities (WI) and the auxiliary equations. As understood ([16]), we have to analyse whether these breakings can be compensated by appropriate finite non-invariant counterterms. These are constrained by power counting and by the system of consistency equations induced by the algebraic properties of the functional operators which generate the required symmetries on field space. In particular we search for new possible anomalies and find that the only possible anomalies are the usual ABJ anomaly [1] which appears both in the ST Identities and in the Ward Identities.

The outline of the paper is as follows. In section 2 we write down the classical Action and introduce the Background Field and the Background Gauge Transformations. Then we give a little account of notation underlining the differences between the conventional

(where the Background Field appears as an infinite collection of gauge parameters not fixed on a particular classical configuration [15]) and the gauge-invariant effective action (where the background field is set equal to the vacuum expectation value of quantum gauge field). In the same section we define the extended BRS symmetry for this model and implement it together with all constraints characterising the tree level action in a system of functional equations. Section 3 is devoted to the extension of previous framework to the quantum regime, (*i.e.* the replacement of the tree level action with the effective action), and to the analysis of the completeness and integrability of constraints. Section 4 is dedicated to the study of the stability of the tree level model and particular care is devoted to the stability of the splitting of gauge field in quantum and classical (background) parts. In section 5 and in its subsections we describe the structure of breakings of functional identities induced by renormalization and we derive the consistency equations to which they must be satisfied by the breaking terms of the ST. Then section 6 is completely devoted to the solution of the consistency equations by using cohomology techniques and the Hodge theorem. In section 7 we check the Ward Identities of Background Gauge Transformations.

2 The Background Field

Let us consider the classical field $V_\mu^a(x)$ belonging to the adjoint representation of Lie algebra of the unitary group $SU(N)$ and gauge-transforming as:

$$V_\mu^a(x) \rightarrow V_\mu^a(x) + \widetilde{\nabla}_\mu^{ab} \lambda_b(x) \quad (1)$$

where $\lambda^a(x)$ is the local infinitesimal parameter and $\widetilde{\nabla}_\mu^{ab}$ is the covariant derivative with respect to $V_\mu^a(x)$.

Unlike the usual point of view we do not take the splitting of the gauge vector field $A_\mu^a(x)$ into Background field and Quantum field in *a priori* manner, but we will obtain that as result from the symmetry imposition on the classical model. Nevertheless we shall call the difference $Q_\mu^a = A_\mu^a - V_\mu^a$ Quantum gauge field from the beginning.

The Yang-Mills classical action for the gauge field $A_\mu^a(x)$ reads:

$$\Gamma^{YM}(A) = \int d^4x \left(-\frac{1}{4g^2} F_{\mu\nu}^a F_a^{\mu\nu} \right) \quad (2)$$

where $F_{\mu\nu}^a(A)$ indicates the usual field strength tensor:

$$F_{\mu\nu}^a(A) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^a_{bc} A_\mu^b A_\nu^c, \quad (3)$$

As it is well known to quantize the model one has to break gauge invariance by choosing a gauge-fixing term and, under the prescriptions of the corresponding BRS symmetry [7], one introduces the ghost terms. We classify as Quantum Gauge Transformations (QGT) those which take only the dynamical degrees of freedom (A_μ^a) in account and leave unmodified the classical background field (V_μ^a):

$$\begin{cases} \delta A_\mu^a = \nabla_\mu^{ab} \lambda_b \\ \delta V_\mu^a = 0 \end{cases} \quad (4)$$

On the contrary, we will call Background Gauge Transformations (BGT) those where the background field transforms under (1):

$$\begin{cases} \delta A_\mu^a = \nabla_\mu^{ab} \lambda_b \\ \delta V_\mu^a = \widetilde{\nabla}_\mu^{ab} \lambda_b \end{cases} \quad (5)$$

so that the difference Q_μ^a is a covariant vector: $\delta Q_\mu^a = f^a{}_{bc} \lambda_b Q_\mu^c$

The prescription of Background Field Method (BFM) ([10]) tells us how to choose a background-field-dependent gauge-fixing term $\Gamma^{g.f.}(A, V)$ in order to break the Quantum Gauge Transformations (4) but keeping the modified action:

$$\Gamma^{YM}(A) \rightarrow \Gamma_0(A, V) = \Gamma^{YM}(A) + \Gamma^{g.f.}(A, V) \quad (6)$$

invariant under Background Gauge Transformations (5).

The roles of background field and of quantum field become manifest upon analysing their transformation properties under (4) and (5). With respect the former the background field stays unchanged and. Therefore it does not contribute to the conserved current associated to the rigid symmetry and coupled to the vector field A_μ^a . The latter transformations will be implemented in the quantum theory by BRS symmetry. Under (5) the field $Q_\mu^a(x)$ behaves as a vector of the adjoint representation and it contributes to the conserved current of background symmetry for which the background field V_μ^a acts as a classical external source.

We express this last symmetry of the action $\Gamma_0(A, V)$ in the functional language by the Ward Identities (WI):

$$W^a(x) \Gamma_0(A, V) \equiv - \left(\nabla_\mu^{ab} \frac{\delta}{\delta A_\mu^b(x)} + \widetilde{\nabla}_\mu^{ab} \frac{\delta}{\delta V_\mu^b(x)} \right) \Gamma_0(A, V) = 0. \quad (7)$$

The local functional:

$$\tilde{\Gamma}_0(A) = \Gamma_0(A, V)|_{A=V} \quad (8)$$

then satisfies the following Ward Identities:

$$[W^a(x) \Gamma_0(A, V)]_{A=V} = - \nabla_\mu^{ab} \frac{\delta}{\delta A_\mu^b(x)} \tilde{\Gamma}_0(A) = 0. \quad (9)$$

Now we are ready to make practical our preliminary considerations. We extend the field space of the theory introducing the ghost fields $\omega^a(x)$, $\bar{\omega}^a(x)$ which have naive dimension one and carry Faddeev-Popov $Q_{\Phi\Pi}$ charge 1 and -1 respectively. We also introduce the Nakanishi-Launtrup Lagrange multiplier $b^a(x)$ ([7], [8]) with dimension two and with no $Q_{\Phi\Pi}$ charge. Each of these fields transforms as a vector of the adjoint representation under BGT.

On this set of fields we can define the conventional BRS transformations by

$$\begin{cases} sA_\mu^a = \nabla_\mu^{ab} \omega_b \\ s\omega^a = \frac{1}{2} f^a{}_{bc} \omega^b \omega^c \\ s\bar{\omega}^a = b^a \quad sb^a = 0 \end{cases} \quad (10)$$

where we choose to indicate with s the generator of BRS.

Then, as usual [7], we complete the set of fields by introducing the external sources $\gamma_\mu^a(x), \zeta^a(x)$ (respectively with $Q_{\Phi\Pi}$ charge -1 and -2 and both with dimension two) coupled respectively to BRS variations of the gauge field $A_\mu^a(x)$ and ghost $\omega^a(x)$. Again, these latter fields are covariant under BGT, invariant under BRS transformations and allow us to translate the non linear BRS symmetry of the model into the Slavnov-Taylor (ST) identities:

$$s(\Gamma_0) = \int d^4x \left[\frac{\delta\Gamma_0}{\delta\gamma_\mu^a} \frac{\delta\Gamma_0}{\delta A_\mu^a} + \frac{\delta\Gamma_0}{\delta\zeta_a} \frac{\delta\Gamma_0}{\delta\omega^a} + b^a \frac{\delta\Gamma_0}{\delta\bar{\omega}^a} \right] = 0 \quad (11)$$

In the same way we can introduce the Ward-Takahashi Identities by means of the generator of BGT $W(\lambda)$:

$$W(\lambda)\Gamma_0[A, V, \omega, \bar{\omega}, b, \gamma, \zeta] \equiv \int d^4x \left[(\nabla\lambda)_\mu^b \frac{\delta\Gamma_0}{\delta A_\mu^b} + (\bar{\nabla}\lambda)_\mu^b \frac{\delta\Gamma_0}{\delta V_\mu^b} + f^a{}_{bc} \lambda^b \Phi^c \frac{\delta\Gamma_0}{\delta\Phi^a} \right] = 0 \quad (12)$$

for which $\Phi^a(x)$ collects all covariant fields $\omega^a(x)$, $\bar{\omega}^a(x)$, $b^a(x)$, $\gamma_\mu^a(x)$ and $\zeta^a(x)$. It is easy to note that the two set of transformations, BRS and BGT commute, so we have the possibility to build a model invariant under both.

Now we can write down our choice of gauge-fixing by using directly the functional equation for the Lagrange multiplier b^a :

$$\int d^4x \beta_a \left[\frac{\delta\Gamma_0}{\delta b^a} + \bar{\nabla}_\mu^{ab} (A_b^\mu - V_b^\mu) - \alpha b^a \right] = 0 \quad (13)$$

where $\beta_a(x)$ is an arbitrary test function covariant under BGT and α is the gauge parameter.

Because of the nilpotence of BRS transformations we build the quantizable version of the theory by adding the BRS variation of an integrated polynomial in the fields to the invariant action $\Gamma^{YM}[A]$ (2):

$$\Gamma_0[A, V, \omega, \bar{\omega}, b, \gamma, \zeta] = \Gamma^{YM}[A] + s \int d^4x \left[\bar{\omega}_a \left(\bar{\nabla}_\mu^{ab} (A - V)_b^\mu + \frac{\alpha}{2} b^a \right) + \gamma_\mu^a A_\mu^a + \zeta^a \omega_a \right] \quad (14)$$

In this way the dependence of background field is confined in the gauge-fixing and ghost terms (this feature suggests that physical observables of this theory are independent on the background field [11]). To implement this feature in the functional language we take the derivative of (14) with respect to the background field $V_\mu^a(x)$, obtaining:

$$\frac{\delta\Gamma_0}{\delta V_\mu^a} = s \left(\bar{\nabla}_\mu^{ab} \bar{\omega}_b \right) \quad (15)$$

In the second member of (15) the BRS variation of a composite operator appears. Then, to freely translate this equation in functional form, we have to couple to it a new anticommuting field $\Omega_\mu^a(x)$ with dimension two, positive $Q_{\Phi\Pi}$ charge and invariant under BRS:

$$\Gamma_0 \rightarrow \Gamma_0 + \int d^4x \left(\Omega_\mu^a \bar{\nabla}_\mu^{ab} \bar{\omega}_b \right) \quad (16)$$

thus we can lead back the background field equation (15) to a generalised ST identities:

$$s(\Gamma_0) = - \int d^4x \left(\Omega_\mu^a s \nabla_\mu^{ab} \bar{\omega}_b \right) = - \int d^4x \left(\Omega_\mu^a \frac{\delta \Gamma_0}{\delta V_\mu^a} \right) \quad (17)$$

by differentiating with respect to the field $\Omega_\mu^a(x)$ and set it to zero. As we will see in the following, the background field equation (15) is fundamental to control the splitting for the gauge field into background gauge field and the quantum one.

It is immediate to note that the previous equation (17) can be simply obtained extending the BRS transformations (10) to the background field:

$$\begin{cases} sV_\mu^a = \Omega_\mu^a \\ s\Omega_\mu^a = 0 \end{cases} \quad (18)$$

here Ω_μ^a assumes the role of a ghost for the background field, and then, differentiating the (17) with respect to the ghost Ω_μ^a we get:

$$\frac{\delta \Gamma_0}{\delta V_\mu^a} = s \left[\frac{\delta \Gamma_0}{\delta \Omega_\mu^a} \right] \quad (19)$$

i.e. the dependence of the theory on the background field is completely confined to a BRS variation.

3 Quantum theory and Functional Identities

We now pass to the quantum theory, defining the generating functional $Z[\bar{J}, \bar{\eta}, V]$ by means of the Feynman integral:

$$Z[\bar{J}, \bar{\eta}, V] = \int \mathcal{D}(A) \mathcal{D}(\omega) \mathcal{D}(\bar{\omega}) \mathcal{D}(b) e^{(i\Gamma_0[\Psi, V, \bar{\eta}] + S.T.)} \quad (20)$$

where we indicate with S.T. the source terms and $\bar{J} = \{J_\mu^a, \bar{\xi}^a, \xi^a, Y^a\}$ collects all the sources for the quantum fields $\Psi = \{A^a, \omega^a, \bar{\omega}^a, b^a\}$ and $\bar{\eta} = \{\gamma_\mu^a(x), \zeta^a(x), \Omega_\mu^a(x)\}$ for the external fields. We explicitly render the dependence of the generating functional $Z[\bar{J}, \bar{\eta}, V]$ on the background field distinguishing the sources \bar{J} from the external fields $\bar{\eta}$. We define the effective action by the Legendre transformation [3]:

$$\Gamma[\Psi, \bar{\eta}, V]_{\Psi = \frac{\delta Z_c}{\delta \bar{J}}} = Z_c[\bar{J}, \bar{\eta}, V,] - \int d^4x (J_\mu^a A_\mu^a + \bar{\omega}^a \xi_a + \bar{\xi}^a \omega_a + b^a Y_a) \quad (21)$$

We want to point out that the derivation of the following equations for generating functional is pure formal because we did not keep in account the unavoidable regularization procedure. For the moment we just suppose to use regularised quantities preserving all sorts of symmetry of our model, although, as known in the Standard Model, no regularization procedure actually does it. We postpone the study of the effects of regularization of Feynman integrals in the next section and here we simply translate the Lagrange multiplier

equation, the WT Identities and the ST Identities in terms of the generating functional (20).

We find for the Lagrange multiplier equation (13) the following relation:

$$\begin{aligned} E(\beta)Z[\bar{J}, \bar{\eta}, V] &\equiv \\ &\equiv \int d^4x \beta_a(x) \left[\widetilde{\nabla}_\mu^{ab} \left(-i \frac{\delta}{\delta J_\mu^b(x)} - V_\mu^b(x) \right) - i\alpha \frac{\delta}{\delta Y^a(x)} + Y^a(x) \right] Z[\bar{J}, \bar{\eta}, V] = 0. \end{aligned} \quad (22)$$

In the same way we translate the ST identities (17) into the linear equation:

$$\mathcal{S}Z[\bar{J}, \bar{\eta}, V] \equiv \int d^4x \left[J_a^\mu \frac{\delta}{\delta \gamma_\mu^a(x)} + \Omega_a^\mu \frac{\delta}{\delta V_\mu^a(x)} - \bar{\xi}_a \frac{\delta}{\delta \zeta^a(x)} - \xi_a \frac{\delta}{\delta Y^a(x)} \right] Z[\bar{J}, \bar{\eta}, V] = 0. \quad (23)$$

Due to nilpotence of BRS transformations and commutation properties of the external field $\Omega_\mu^a(x)$ and of background field $V_\mu^a(x)$, the functional operator \mathcal{S} is nilpotent.

As usual [7] we recover the Faddeev-Popov field equation calculating explicitly the commutator between the Slavnov-Taylor operator \mathcal{S} and the operator $E(\beta)$ defined in (22):

$$\begin{aligned} \Sigma(\beta)Z[\bar{J}, \bar{\eta}, V] &\equiv [\mathcal{S}, E(\beta)] Z[\bar{J}, \bar{\eta}, V] = \\ &= \int d^4x \beta_a(x) \left[\left(\partial^\mu \Omega_\mu^a - i f^a_{bc} \Omega_\mu^b \frac{\delta}{\delta J_\mu^c(x)} \right) + \xi^a + i \widetilde{\nabla}_\mu^{ab} \frac{\delta}{\delta \gamma_\mu^b(x)} \right] Z[\bar{J}, \bar{\eta}, V] = 0 \end{aligned} \quad (24)$$

Finally taking into account the invariance properties of the classical action Γ_0 and the variation of source terms in the generating functional (20), we find the corresponding WT Identities for $Z[\bar{J}, \bar{\eta}, V]$:

$$\begin{aligned} W(\lambda)Z[\bar{J}, \bar{\eta}, V] &\equiv \\ &\equiv \int d^4x \left[J_a^\mu \left(\partial_\mu \lambda^a + i f_{bc}^a \lambda^b \frac{\delta}{\delta J_\mu^c(x)} \right) + \widetilde{\nabla}_\mu^{ab} \lambda_b \frac{\delta}{\delta V_\mu^a(x)} + i (f_{bc}^a \lambda_a) \Phi^b \frac{\delta}{\delta \Phi^c(x)} \right] Z[\bar{J}, \bar{\eta}, V] = 0 \end{aligned} \quad (25)$$

where we collected all covariant fields $\omega^a, \bar{\omega}^a, \dots$ under a single symbol $\Phi^a(x)$.

The complete algebra of functional operators $\{\mathcal{S}, W(\beta), \Sigma(\beta), E(\beta)\}$ is given by the commutators:

$$[W(\lambda), E(\beta)] = E(\lambda \wedge \beta) \quad [W(\lambda), W(\beta)] = W(\lambda \wedge \beta) \quad [E(\lambda), \Sigma(\beta)] = 0 \quad (26)$$

$$[\mathcal{S}, E(\beta)] = \Sigma(\beta) \quad \{\mathcal{S}, \Sigma(\beta)\} = 0 \quad [\mathcal{S}, W(\beta)] = 0 \quad [W(\lambda), \Sigma(\beta)] = \Sigma(\lambda \wedge \beta) \quad (27)$$

i.e. they belong to an involutive algebra and, by Fröbenius theorem, they generate a completely integrable differential system.

4 Stability of classical model

We now come back to the functional equations for the classical action Γ_0 . In the previous sections we have defined a classical action and we have described the set of constraints to which this must satisfy. We now want to verify that the most general solution of these constraints is our starting classical action up to possible multiplicative field renormalizations. Then in the following we shall consider the local functional Γ_0 unknown and we proceed to solve completely the set of functional equations.

The first step toward a simplification of the cumbersome set of functional equations satisfied by Γ_0 consists in substituting it with the new modified version:

$$\hat{\Gamma}_0 = \Gamma_0 - \int d^4x \left[b^a \widetilde{\nabla}_\mu^{ab} (A_b^\mu - V_a^\mu) + \frac{\alpha}{2} b^a b_a + \Omega_a^\mu \nabla_\mu^{ab} \bar{\omega}_b \right] \quad (28)$$

We subtracted the gauge-fixing terms because of their non-renormalization properties and we also chose to subtract away the Ω -dependent terms.

Then it is easy to verify that the new effective action $\hat{\Gamma}_0$ satisfies the system of equations:

$$\left\{ \begin{array}{l} \int d^4x \beta^a \frac{\delta \hat{\Gamma}_0}{\delta b^a} = 0 \\ \int d^4x \beta_a \left(\frac{\delta \hat{\Gamma}_0}{\delta \bar{\omega}^a} - \widetilde{\nabla}_\mu^{ab} \frac{\delta \hat{\Gamma}_0}{\delta \gamma_\mu^b} \right) = 0 \\ \int d^4x \left[\frac{\delta \hat{\Gamma}_0}{\delta A_\mu^a} \frac{\delta \hat{\Gamma}_0}{\delta \gamma_\mu^a} + \frac{\delta \hat{\Gamma}_0}{\delta \omega^a} \frac{\delta \hat{\Gamma}_0}{\delta \zeta_a} + \Omega_\mu^a \left(\frac{\delta \hat{\Gamma}_0}{\delta V_\mu^a} - f_{abc} \bar{\omega}^b \frac{\delta \hat{\Gamma}_0}{\delta \gamma_\mu^c} \right) \right] = 0 \\ \int d^4x \left[\nabla_\mu^{ab} \lambda^b \frac{\delta \hat{\Gamma}_0}{\delta A_\mu^c} + \widetilde{\nabla}_\mu^{ab} \lambda_b \frac{\delta \hat{\Gamma}_0}{\delta V_\mu^a} + (f^a{}_{bc} \lambda_a) \Phi^b \frac{\delta \hat{\Gamma}_0}{\delta \Phi^c} \right] (x) = 0 \end{array} \right. \quad (29)$$

The first equation shows that the new action $\hat{\Gamma}_0$ does not depend upon the b^a field; the second one gives an important constraint on the antighost field $\bar{\omega}^a$, forcing it to appear only in the linear combination:

$$\hat{\gamma}_\mu^a = \gamma_\mu^a - \widetilde{\nabla}_\mu^{ab} \bar{\omega}_b \quad (30)$$

Since every change from Γ_0 to $\hat{\Gamma}_0$ is gauge invariant no breaking term appears in the WT Identities (25). On the other hand we have to modify slightly the ST identities.

Because $\hat{\gamma}_\mu^a(x)$ is of dimension two and has charge $Q_{\Phi\Pi}=-1$ the action $\hat{\Gamma}_0$ may only depend linearly upon it. Then we can rewrite the ST identities (29) in the form:

$$\mathcal{D}_{\hat{\Gamma}_0} \hat{\Gamma}_0 \equiv \int d^4x \left[\frac{\delta \hat{\Gamma}_0}{\delta A_\mu^a} \frac{\delta \hat{\Gamma}_0}{\delta \hat{\gamma}_\mu^a} + \frac{\delta \hat{\Gamma}_0}{\delta \omega^a} \frac{\delta \hat{\Gamma}_0}{\delta \zeta_a} + \Omega_\mu^a \frac{\delta \hat{\Gamma}_0}{\delta V_\mu^a} \right] = 0 \quad (31)$$

In order to simplify the analysis of ST identities we introduce the linear operator:

$$\mathcal{F}_{\hat{\Gamma}_0} \equiv \int d^4x \left[\frac{\delta \hat{\Gamma}_0}{\delta A_\mu^a} \frac{\delta}{\delta \hat{\gamma}_\mu^a} + \frac{\delta \hat{\Gamma}_0}{\delta \gamma_\mu^a} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \hat{\Gamma}_0}{\delta \omega^a} \frac{\delta}{\delta \zeta_a} + \frac{\delta \hat{\Gamma}_0}{\delta \zeta_a} \frac{\delta}{\delta \omega^a} + \Omega_\mu^a \frac{\delta}{\delta V_\mu^a} \right] \quad (32)$$

The ST identities themselves imply the consistency equations

$$\mathcal{F}_{\hat{\Gamma}_0} \mathcal{D}_{\hat{\Gamma}_0} \hat{\Gamma}_0 = 0 \quad (33)$$

and the commutability of BGT with BRS transformations may be rewritten in the form:

$$\mathcal{F}_{\hat{\Gamma}_0} W(\lambda) - W(\lambda) \mathcal{D}_{\hat{\Gamma}_0} = 0 \quad (34)$$

We now can complete the check of stability on the classical model by finding the general solution of functional equations (29) for the functional $\hat{\Gamma}_0$ which depends only on the reduced set of fields $A_\mu^a, \hat{\gamma}_\mu^a, \omega^a, \zeta^a$ and Ω_μ^a .

We start analysing the Ω_μ^a dependent part of the action. Since this field is characterized by $Q_{\Phi\Pi}$ charge +1 and naive dimension 2, the only gauge invariant integrated polynomial which respects the quantum numbers of the functional $\hat{\Gamma}_0$ is forced to be

$$\Delta = u \int d^4x (\Omega_\mu^a \hat{\gamma}_a^\mu) \quad (35)$$

where u is a constant.

Redefining a new action with

$$\bar{\Gamma}_0 = \hat{\Gamma}_0 - u \int d^4x (\Omega_\mu^a \hat{\gamma}_a^\mu) \quad (36)$$

and exploiting the independence of new action $\bar{\Gamma}_0$ on the Ω_μ^a field, the ST identities (31) split into the system of functional equations:

$$\begin{cases} \int d^4x \left[\frac{\delta \bar{\Gamma}_0}{\delta A_\mu^a} \frac{\delta \bar{\Gamma}_0}{\delta \hat{\gamma}_\mu^a} + \frac{\delta \bar{\Gamma}_0}{\delta \omega^a} \frac{\delta \bar{\Gamma}_0}{\delta \zeta_a} \right] = 0 \\ \left(\frac{\delta \bar{\Gamma}_0}{\delta V_\mu^a} - u \frac{\delta \bar{\Gamma}_0}{\delta A_\mu^a} \right) = 0. \end{cases} \quad (37)$$

From the second equation we deduce that the vector fields $A_\mu^a(x)$ and $V_\mu^a(x)$ appear in the solution of our problem only through the linear combination:

$$\hat{A}_\mu^a = A_\mu^a + u V_\mu^a \quad (38)$$

If we substitute the field $\hat{A}_\mu^a(x)$ in place of $A_\mu^a(x)$, the first equation of the system (37) coincides with the usual ST identities, for which the solution is well-known [7]:

$$\bar{\Gamma}_0[\hat{A}, \omega, \zeta, \hat{\gamma}] = \int d^4x \left[-\frac{1}{4g^2} F_a^{\mu\nu} F_{\mu\nu}^a + \frac{z}{2} f_{abc} \zeta^a \omega^b \omega^c + (zZ) \hat{\gamma}_\mu^a (\partial_\mu \omega^a - Z^{-1} f_{abc} \hat{A}_\mu^b \omega^c) \right] \quad (39)$$

where

$$F_{\mu\nu}^a(\hat{A}) = Z^{-1} (\partial_\mu \hat{A}_\nu^a - \partial_\nu \hat{A}_\mu^a - Z^{-1} f_{bc}^a \hat{A}_\mu^b \hat{A}_\nu^c) \quad (40)$$

and the constants g , z and Z are the only free parameters.

As it stands the linear combination (38) is not background gauge invariant, but requiring this property we arrive to:

$$W(\lambda)(Z^{-1}\hat{A}_\mu^a) = Z^{-1}(1+u)\partial_\mu\lambda^a - f^a_{bc}\lambda^b Z^{-1}\hat{A}_\mu^c \quad (41)$$

i.e. $Z = 1 + u$. Rewriting this result in terms of Q_μ^a and V_μ^a we obtain

$$Z^{-1}\hat{A}_\mu^a = Z^{-1}(A + uV)_\mu^a = Z^{-1}Q_\mu^a + V \quad (42)$$

from which one immediately sees that only the Q_μ^a field is multiplicative renormalized. Then for a Yang-Mills theory quantized with background gauge the splitting between quantum field Q_μ^a (it is now manifest why we called it Quantum Field) and the background field V_μ^a is obtained from the constraints of symmetry imposed on it. This also shows that the decomposition is stable and no radiative correction may mix the quantum part with classical one. We wish to stress that the field renormalization $\hat{A} \rightarrow Z^{-1}\hat{A}$ is the usual field renormalization of Yang-Mills theory.

We conclude this part with some considerations about the normalization conditions. The manifest gauge invariance of the effective action (8) implies the counter-terms dependence only upon gauge-independent “physical renormalization” *i.e.* on charge renormalization. Then we have no need to specify the normalization conditions for quantum fields *i.e.* for the wave function renormalization of quantum gauge fields and of the ghost fields, or in other words, the physical quantities are insensitive on these intermediate normalization conditions. This aspect of BFM will be very useful in the SM in the spontaneously symmetry broken phase because of large amount of gauge-dependent (“non physical”) renormalization constants especially if one decides to work in a covariant gauge.

In the parametrization of the tree level action given in ([19]) the arbitrariness is confined by constraints to the wave function renormalizations Z for gauge fields and z for ghost fields and in an overall renormalization of the action. In fact in this parametrization it is easy (by means of an enlarged set of BRS transformations comprising the BRS variation for the gauge parameter) to show the gauge-independence of the overall renormalization. On the contrary the two constant Z, z result to be gauge-dependent parameters, and so are the corresponding anomalous dimensions γ_A, γ_c .

This parametrization is particular advantageous in the BFM because here the overall constant coincides with the inverse of the square of the coupling constant, so that the overall renormalization for the effective action $\Gamma[A, V]$ becomes the overall renormalization for the gauge invariant effective action $\tilde{\Gamma}[A]$. Gauge invariance then tell us that this renormalization is the charge renormalization.

Then we can fix the following normalization condition:

$$\begin{aligned} \tilde{\Gamma}_{A_\mu^a A_\nu^b}(p) &= \delta^{ab} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Lambda_A(p) \\ \frac{\partial}{\partial p^2} \Lambda_A(p) \big|_{p^2=\kappa^2} &= \frac{-1}{g^2} \end{aligned} \quad (43)$$

where κ^2 is the (euclidean) normalization point. This choice of κ^2 is irrelevant at the tree level, and in any case under complete control by means of Callan-Symanzik equation.

The particular parametrization chosen here for the BFM-version of pure Yang-Mills implies immediately the Kallosh theorem [20]. In fact the independence of the overall renormalization from gauge parameter is equivalent to that of the coupling constant renormalization, *i.e.* the Kallosh theorem. This fact implies also the independence from a fixed gauge parameter for the normalization condition (43) (compare this with the corresponding normalization condition in [19]).

5 Renormalization

5.1 Consistency equations

Since the symmetries characterising the theory are acceptable for any purpose, including for instance the background gauge invariance of the quantum action $\hat{\Gamma}$ and the determination of counter terms, it is necessary that they survive the quantization process. In this section our intent is to describe how the system of previous equations for the functional $\hat{\Gamma}$ will be modified by the breaking terms induced by radiative corrections and by the subtraction procedure and we will verify that the system can be restored order by order by an appropriate choice of counter terms.

The main tool for this investigation is the Quantum Action Principle (QAP) [2], proved in the BPHZL renormalization scheme [6], [5], which states that the functional equations of the model get broken at the quantum level by local integrated insertions. Moreover, as it is usual [7] in the standard algebraic procedure, the commutation properties (26) and (27) of functional operators and the features of local insertions allow us to derive a set of consistency equations to which these breaking terms have to satisfy.

Solving the set of consistency equations we individuate all possible breaking terms. In general, at this point two situations could happen: either all these breaking terms are variations of non symmetric counter terms for the tree level action, *i.e.* they can be compensated, or there is at least a non-compensable breaking terms; in this latter case the corresponding broken equation gets an anomalous term or, on the other hands, the symmetry described by that equation is not a symmetry any longer at the quantum level.

In the present section we use the notation $\hat{\Gamma}$ to indicate 1PI generating functional, the symbol $\hat{\Gamma}_0$ for the corresponding tree approximated action, and $\hat{\Gamma}_{Eff}$ for the effective action where the coefficients of integrated field polynomial of $\hat{\Gamma}_0$ are replaced by formal power series of \hbar . The last functional $\hat{\Gamma}_{Eff}$ deals with order by order counter terms that implement the normalization conditions and symmetries at the quantum level.

In order to protect our calculations from infrared divergences induced by a zero momentum subtractions of BPHZ scheme, we adopt the modification introduced by J.Lowenstein [5]:

$$\begin{aligned} \hat{\Gamma}_0(Q, V, \omega, \hat{\gamma}, \zeta, \Omega) &\rightarrow \\ \rightarrow \hat{\Gamma}_0(Q, V, \omega, \hat{\gamma}, \zeta, \Omega; s) &\equiv \hat{\Gamma}_0(Q, V, \omega, \hat{\gamma}, \zeta, \Omega) + \mu^2(s-1)^2 \int d^4x \left(\frac{1}{2} Q_\mu^a Q_a^\mu + \bar{\omega}^a \omega_a \right) \end{aligned} \quad (44)$$

The propagators for the massless fields Q_μ^a , ω^a e $\bar{\omega}^a$ acquire a parameter s dependent mass; this parameter s varies between zero and one and will play the role of an additional subtraction variable (like an external momentum p). At the end of calculations it is to be put equal to one for all massless fields. At $s \neq 1$ such mass term will describe a massive field (with off-shell normalization conditions [5]).

Nevertheless we have to note that the ST identities (11) for the classical action $\hat{\Gamma}_0$ is spoiled by those new terms because they are not BRS invariant, and, in the same way, also the equation of motion for the ghost field $\bar{\omega}^a(x)$ is modified. The presence of these breaking terms does not alter the content of BRS symmetry because they are only soft breaking terms and they have to be treated by the technique of the reference [4], [1] and [16]. By the way we will find some other $(s-1)$ -dependent terms during the building of counter terms, but we have to keep in mind the limit $s \rightarrow 1$ as final act of renormalization procedure [4] and so we have not to show their explicit structure. On the other way, we note that the mass terms are background gauge invariant, and no breaking term occurs for the corresponding WT Identities.

Also for ghost equation of motion we meet the same difficulties of ST identities because of regularization of massless propagators, and the basic simple constrain given by this equation is lost. Nevertheless, since we are interested into the theory for $s = 1$, we assume the dependence of the functional $\hat{\Gamma}$ on fields $\gamma_\mu^a(x)$ and $\bar{\omega}_b(x)$ is only through the linear combination:

$$\hat{\gamma}_\mu^a = \gamma_\mu^a - \widetilde{\nabla}_\mu^{ab} \bar{\omega}_b \quad (45)$$

which is correct up to $(s - 1)$ corrections. (See [18] for a more detailed discussion).

We now can complete the translation of our functional identities in the BPHZL contest obtaining:

$$\left\{ \begin{array}{l} \int d^4x \left[\frac{\delta \hat{\Gamma}}{\delta A_\mu^a} \frac{\delta \hat{\Gamma}}{\delta \hat{\gamma}_a^\mu} + \frac{\delta \hat{\Gamma}}{\delta \omega^a} \frac{\delta \hat{\Gamma}}{\delta \zeta_a} + \Omega_\mu^a \frac{\delta \hat{\Gamma}}{\delta V_\mu^a} \right] = \Delta_{ST} \hat{\Gamma} \\ \int d^4x \left[\widetilde{\nabla}_\mu^{ab} \lambda_b \frac{\delta \hat{\Gamma}}{\delta V_\mu^a} + \nabla_\mu^{ab} \lambda_b \frac{\delta \hat{\Gamma}}{\delta A_\mu^a} + f^a{}_{bc} \lambda_a \Phi_b \frac{\delta \hat{\Gamma}}{\delta \Phi^c} \right] = \Delta_W \hat{\Gamma} \end{array} \right. \quad (46)$$

These equations are different from the classical ones (11) and (12) because of the breaking terms $\Delta_{ST} \hat{\Gamma}$ and $\Delta_W(\lambda) \hat{\Gamma}$ which sum up the non invariant contributions from loop computations. Following the notations of [4] or [6] we can make the UV, IR and $(s - 1)$ indices of subtraction for the breakings Δ_{ST} e $\Delta_W(\lambda)$ explicit:

$$\begin{aligned} \Delta_{ST} \hat{\Gamma} &= \int d^4x N_{5,2}^5 [Q_{ST}(x)] \hat{\Gamma} \\ \Delta_W(\lambda) \hat{\Gamma} &= \int d^4x \lambda_a N_{4,2}^4 [Q_W^a(x)] \hat{\Gamma} \end{aligned} \quad (47)$$

where the first local polynomial $Q_{ST}(x)$ is scalar under Poincaré transformations, invariant under rigid gauge transformations and carries the charge $Q_{\Phi\Pi}=1$; the second one: $Q_W^a(x)$ is also scalar under Poincaré transformations, carries an index of adjoint representation of rigid gauge transformations and no $Q_{\Phi\Pi}$ charge. These quantum numbers assigned to them

classify completely the algebraic structure of the breaking terms. Their coefficients have to be fixed on normalization conditions.

We now proceed by induction technique supposing that breakings of the identities (46) were reabsorbed up to the \hbar^{n-1} -order of perturbation theory and, using the properties of Zimmermann's normal products, we can separate \hbar^n -order contributions from those of higher order:

$$\begin{aligned}\Delta_{ST}\hat{\Gamma} &= \hbar^n \int d^4x P_{ST}^{(n)}(x) + O(\hbar^{n+1}) \\ \Delta_W(\lambda)\hat{\Gamma} &= \hbar^n \int d^4x \lambda_a P_W^{(n)a}(x) + O(\hbar^{n+1})\end{aligned}\tag{48}$$

From these definitions we deduce the consistency equations to which the $\Delta_{ST}\hat{\Gamma}$ and $\Delta_W(\lambda)\hat{\Gamma}$ have to satisfy. We rewrite the system (46) in the synthetic notation:

$$\begin{cases} \mathcal{D}_{\hat{\Gamma}}\hat{\Gamma} = \hbar^n \Delta_{ST}^{(n)} + O(\hbar^{n+1}) \\ W(\lambda)\hat{\Gamma} = \hbar^n \Delta_W^{(n)}(\lambda) + O(\hbar^{n+1}) \end{cases}\tag{49}$$

and recall the nilpotence or the commutation properties of functional operators $\mathcal{D}_{\hat{\Gamma}}$, $\mathcal{F}_{\hat{\Gamma}}$ and $W(\lambda)$ (26) and (27):

$$\mathcal{F}_{\hat{\Gamma}}\mathcal{D}_{\hat{\Gamma}}\hat{\Gamma} = 0, \quad \mathcal{F}_{\hat{\Gamma}}W(\lambda) - W(\lambda)\mathcal{D}_{\hat{\Gamma}} = 0, \quad [W(\lambda), W(\beta)] = W(\lambda \wedge \beta)\tag{50}$$

where $\mathcal{F}_{\hat{\Gamma}}$ is the linear ST operator of the previous section.

If we act with the operator $\mathcal{F}_{\hat{\Gamma}}$ on the left upon the first equations of (49) and by (50), we obtain:

$$\mathcal{F}_{\hat{\Gamma}}\mathcal{D}_{\hat{\Gamma}}\hat{\Gamma} = \hbar^n \mathcal{F}_{\hat{\Gamma}}\Delta_{ST}^{(n)} + O(\hbar^{n+1}) = 0\tag{51}$$

from which, organising at the best the \hbar^n -order terms, and recalling $\hat{\Gamma} = \hat{\Gamma}_0 + \hbar\hat{\Gamma}_1$, it follows:

$$\hbar^n \mathcal{F}_{\hat{\Gamma}}\Delta_{ST}^{(n)} + O(\hbar^{n+1}) = \hbar^n \mathcal{F}_{\hat{\Gamma}_0}\Delta_{ST}^{(n)} + O(\hbar^{n+1})\tag{52}$$

i.e. the consistency equations:

$$\mathcal{F}_{\hat{\Gamma}_0}\Delta_{ST}^{(n)} = 0\tag{53}$$

On the other hand acting respectively on the first and on the second equation with the operators $W(\lambda)$ and $\mathcal{F}_{\hat{\Gamma}}$, we arrive at:

$$\begin{cases} W(\lambda)\mathcal{D}_{\hat{\Gamma}}\hat{\Gamma} = \hbar^n W(\lambda)\Delta_{ST}^{(n)} + O(\hbar^{n+1}) \\ \mathcal{F}_{\hat{\Gamma}}W(\lambda)\hat{\Gamma} = \hbar^n \mathcal{F}_{\hat{\Gamma}_0}\Delta_W^{(n)}(\lambda) + O(\hbar^{n+1}) \end{cases}\tag{54}$$

and by the (50) at:

$$W(\lambda)\Delta_{ST}^{(n)} - \mathcal{F}_{\hat{\Gamma}_0}\Delta_W^{(n)}(\lambda) = 0\tag{55}$$

up to the order \hbar^n .

Finally the commutation properties for the $W(\lambda)$ operators imply the following consistency equations:

$$W(\lambda)\Delta_W^{(n)}(\beta) - W(\beta)\Delta_W^{(n)}(\lambda) = \Delta_W^{(n)}(\lambda \wedge \beta) \quad (56)$$

which can be identified with the well-known Wess-Zumino conditions [17].

In the next sections we will completely analyse the set of consistency equations (53), (55) and (56). We want to recall that also the Faddeev-Popov ghost equations and the Lagrange multiplier equations are modified by breaking terms but for their analysis we remand the reader to [18] again.

5.2 Consistency equations for ST breakings

To analyse the consistency equations (53) for ST breakings $\Delta_{ST}^{(n)}$ we have to decompose the operators $\mathcal{F}_{\hat{\Gamma}}$ and $\mathcal{D}_{\hat{\Gamma}}$ in simpler ones:

$$\begin{aligned} \mathcal{B}_{\hat{\Gamma}} &= \int d^4x \left(\frac{\delta \hat{\Gamma}}{\delta A_\mu^a} \frac{\delta}{\delta \gamma_a^\mu} + \frac{\delta \hat{\Gamma}}{\delta \zeta^a} \frac{\delta}{\delta \omega_a} \right) & \bar{\mathcal{B}}_{\hat{\Gamma}} &= \int d^4x \left(\frac{\delta \hat{\Gamma}}{\delta \gamma_a^\mu} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \hat{\Gamma}}{\delta \omega_a} \frac{\delta}{\delta \zeta^a} \right) \\ \mathcal{R} &= \int d^4x \left(\Omega_\mu^a \frac{\delta}{\delta V_\mu^a} \right) \end{aligned} \quad (57)$$

by means of them we have:

$$\mathcal{D}_{\hat{\Gamma}} = \mathbf{D}_{\hat{\Gamma}} + \mathcal{R} = \frac{1}{2}(\mathcal{B}_{\hat{\Gamma}} + \bar{\mathcal{B}}_{\hat{\Gamma}}) + \mathcal{R} \quad \mathcal{F}_{\hat{\Gamma}} = 2\mathbf{D}_{\hat{\Gamma}} + \mathcal{R} = (\mathcal{B}_{\hat{\Gamma}} + \bar{\mathcal{B}}_{\hat{\Gamma}} + \mathcal{R}) \quad (58)$$

where the $\hat{\Gamma}$ -dependent and $\hat{\Gamma}$ -independent parts of operators $\mathcal{D}_{\hat{\Gamma}}$ and $\mathcal{F}_{\hat{\Gamma}}$ are splitted.

We now want to show how to build the counter terms. One easily checks that the following algebraic relations hold:

$$\mathcal{B}_{\hat{\Gamma}}\hat{\Gamma} = \bar{\mathcal{B}}_{\hat{\Gamma}}\hat{\Gamma} \quad \mathcal{B}_{\hat{\Gamma}}\Delta = \bar{\mathcal{B}}_{\Delta}\hat{\Gamma} \quad \bar{\mathcal{B}}_{\hat{\Gamma}}\Delta = \mathcal{B}_{\Delta}\hat{\Gamma} \quad (59)$$

and

$$\mathcal{F}_{\hat{\Gamma}}^2 = \mathcal{B}_{\mathcal{D}_{\hat{\Gamma}}\hat{\Gamma}} - \bar{\mathcal{B}}_{\mathcal{D}_{\hat{\Gamma}}\hat{\Gamma}} \quad (60)$$

Hence $\mathcal{F}_{\hat{\Gamma}}$ is nilpotent if the ST identities $\mathcal{D}_{\hat{\Gamma}}\hat{\Gamma} = 0$ hold, as, for instance, for the classical action $\hat{\Gamma}_0$.

If we suppose that there is a local polynomial $Q_0^n(x)$ such as

$$\Delta_0^{(n)} = \int d^4x Q_0^n(x) \quad \Delta_{ST}^{(n)} = \mathcal{F}_{\hat{\Gamma}_0} \Delta_0^{(n)} \quad (61)$$

so that the breaking terms $\Delta_{ST}^{(n)}$ satisfy the consistency equation (53) because of nilpotence of $\mathcal{F}_{\hat{\Gamma}_0}$, then the modified effective action:

$$\bar{\Gamma} \stackrel{def}{=} \hat{\Gamma} - \hbar^n \Delta_0^{(n)} \quad (62)$$

satisfies the following equations:

$$\mathcal{D}_{\bar{\Gamma}}\bar{\Gamma} = \left[\frac{1}{2}(\mathcal{B}_{\bar{\Gamma}} + \bar{\mathcal{B}}_{\bar{\Gamma}}) + \mathcal{R} \right] (\hat{\Gamma} - \hbar^n \Delta_0^{(n)}) = \Delta_{ST}^{(n)} - \hbar^n \mathcal{F}_{\hat{\Gamma}_0} \Delta_0^{(n)} + O(\hbar^{n+1}) = O(\hbar^{n+1}) \quad (63)$$

i. e.

$$\mathcal{D}_{\bar{\Gamma}}\bar{\Gamma} = O(\hbar^{n+1}) \quad (64)$$

and consequently solves our renormalization problem of ST identities. Obviusosly we have to show that our hypothesis (61) is correct, and this is our aim in the following.

We now enter in a more detail analysis of breaking terms Δ_{ST} , and from the beginnings we can split them into two parts respectively dependent and independent on s parameter:

$$\Delta_{ST}^{(n)} = \int d^4x P_{ST}^{(n)}(x) + \sum_{k=1}^2 (s-1)^k \int d^4x R_{ST,k}^{(n)}(x). \quad (65)$$

Here the local polynomials $R_{ST,k}^{(n)}(x)$, $P_{ST}^{(n)}(x)$ are both independent on s parameter. The polynomial $P_{ST}^{(n)}(x)$ has IR and UV degree equal to 5, whereas the $R_{ST,k}^{(n)}(x)$ terms have $d_{R_{ST,k}^{(n)}} = r_{R_{ST,k}^{(n)}} = (5-k)$. Since we are only interest in the s=1 theory, we have not to determine the explicit form of $R_{ST,k}^{(n)}(x)$, but we drive our attention on the first term of right hand side of eq. (65).

One sees that, due to the dimensions and charges of the fields, the ST breaking $\Delta_{ST}^{(n)}$ can be break up into the following two terms:

$$\Delta_{ST}^{(n)} = \int d^4x \left(\Omega_{\mu}^a P_{1a}^{(n)\mu} + P_2^{(n)} \right) \quad (66)$$

where $P_{1a}^{(n)\mu}$, $P_2^{(n)}$ are both functions of fields $\{A_{\mu}^a, V_{\mu}^a, \hat{\gamma}_{\mu}^a, \omega^a, \zeta^a\}$ and by their derivatives.

Recalling that the external field $\Omega_{\mu}^a(x)$ has $Q_{\Phi\Pi} = 1$ and dimension 2, we can easily rewrite the first term of r.h.s. of eq. (65) in the form:

$$\Omega_{\mu}^a P_{1a}^{(n)\mu} = \Lambda_{abc} \Omega_{\mu}^a \hat{\gamma}^{b\mu} \omega^c + \Omega_{\mu}^a \tilde{P}_{1a}^{\mu}(A, V) \quad (67)$$

where the field dependence of \tilde{P}_{1a}^{μ} is pointed out and Λ^{abc} is an invariant tensor under SU(N) group transformations. The first term of (67) is easily compensated by introducing the counter term:

$$\hat{\Gamma}_{\Omega\hat{\gamma}\omega} \equiv \int d^4x \Lambda_{abc} \hat{\gamma}_{\mu}^a \omega^b V_a^{\mu} \quad (68)$$

and observing that the following equation holds:

$$\int d^4x \Lambda_{abc} \Omega_{\mu}^a \hat{\gamma}^{b\mu} \omega^c = \mathcal{R} \hat{\Gamma}_{\Omega\hat{\gamma}\omega}. \quad (69)$$

The expressions (66) and (67) allow us to reduce the consistency equation (53) to a system of functional equations, in fact making explicit $\mathcal{F}_{\hat{\Gamma}_0}$ and $\Delta_{ST}^{(n)}$, we get:

$$\begin{aligned} \mathcal{F}_{\hat{\Gamma}_0} \Delta_{ST}^{(n)} &= 2\mathbf{D}_{\hat{\Gamma}_0} \Delta_{ST}^{(n)} + \mathcal{R} \Delta_{ST}^{(n)} = \\ &= 2\mathbf{D}_{\hat{\Gamma}_0} \int d^4y \left(\Omega_{\mu}^a P_{1a}^{(n)\mu} + P_2^{(n)} \right) + \int d^4x \left[\Omega_{\mu}^a \frac{\delta}{\delta V_{\mu}^a} \right] \int d^4x \left(\Omega_{\mu}^a P_{1a}^{(n)\mu} + P_2^{(n)} \right). \end{aligned} \quad (70)$$

Then collecting the terms with increasing powers of Ω_μ^a , we find:

$$\begin{cases} \mathbf{D}_{\hat{\Gamma}_0} \int d^4x P_2^{(n)} = 0 \\ 2\mathbf{D}_{\hat{\Gamma}_0} \int d^4x \left(\Omega_\mu^a P_{1a}^{(n)\mu} \right) + \mathcal{R} \int d^4x P_2^{(n)} = 0 \\ \int d^4x \left[\Omega_\mu^a \frac{\delta}{\delta V_\mu^a} \right] \int d^4y \Omega_\mu^a P_{1a}^{(n)\mu} = 0 \end{cases} \quad (71)$$

and by (67) the last equation becomes:

$$\int d^4x \Omega_\mu^a(x) \frac{\delta}{\delta V_\mu^a} \int d^4y \Omega_\mu^a \tilde{P}_{1a}^{(n)\mu} = 0 \quad (72)$$

We postpone the study of this equation until the next section, because we now want to study the implications of system (71) for the final result.

We suppose to have been able to show that all Ω_μ^a dependent terms of the ST breakings are compensable by a set of Ω_μ^a -independent counterterms $\{\hat{\Gamma}_i\}$

$$\int d^4x \left(\Omega_\mu^a P_{1a}^{(n)\mu} \right) = \mathcal{R} \sum_i \hat{\Gamma}_i \quad (73)$$

then modifying the effective action by

$$\hat{\Gamma} \rightarrow \bar{\Gamma} \equiv \hat{\Gamma} - \hbar^n \sum_i \hat{\Gamma}_i \quad (74)$$

we obtain the following result:

$$\mathcal{D}_{\bar{\Gamma}} \bar{\Gamma} = (\mathbf{D}_{\bar{\Gamma}} + \mathcal{R})(\hat{\Gamma} - \hbar^n \sum_i \hat{\Gamma}_i) = \hbar^n \left(\int d^4x P_2^{(n)} - \hat{\Delta}_{ST}^{(n)} \right) + O(\hbar^{n+1}) \quad (75)$$

where $\hat{\Delta}_{ST}^{(n)} \equiv 2 \sum_i \mathbf{D}_{\hat{\Gamma}_i} \hat{\Gamma}_0$. We can summarize the previous equations into the following:

$$\mathcal{D}_{\bar{\Gamma}} \bar{\Gamma} = \hbar^n \tilde{\Delta}_{ST}^{(n)} + O(\hbar^{n+1}) \quad (76)$$

where the term $\tilde{\Delta}_{ST}^{(n)}$ collects all Ω -independent contributes coming directly from ST breakings and from $\mathbf{D}_{\hat{\Gamma}_i} \hat{\Gamma}_0$. Thus we deduce that

$$\frac{\delta \tilde{\Delta}_{ST}^{(n)}}{\delta \Omega_\mu^a} = 0. \quad (77)$$

Let us study the equation:

$$\frac{\delta \hat{\Gamma}}{\delta \Omega_\mu^a} = \Delta_\Omega \hat{\Gamma} \quad (78)$$

which describes the dependence of 1PI on Ω_μ^a field, and by the means of QAP, it take in account the breakings from tree level equation:

$$\frac{\delta \hat{\Gamma}_0}{\delta \Omega_\mu^a} = 0 \quad (79)$$

The breaking term $\Delta_\Omega^{(n)}(x)$ at \hbar^n order is a local polynomial with $Q_{\Phi\Pi} = -1$, it is a vector under Lorentz transformations, a vector under global gauge transformations and is characterised by the Zimmermann's degrees of subtraction: $\delta_{\Delta_\Omega} \leq 2$, $\rho_{\Delta_\Omega} \geq 2$, $\deg_{s-1} \Delta_\Omega \geq 2$.

One immediately sees that $\Delta_\Omega^{(n)}$ does not contain powers of $(s-1)$ because of algebraic structure it belongs to; moreover it is apparent that the only possible monomial is:

$$\Delta_{\Omega\mu}^{(n)a} = \Lambda_{\mu\nu}^{ab} \hat{\gamma}_b^\nu \quad (80)$$

that is a linear term in the field $\hat{\gamma}_\mu^a(x)$. This fact removes the necessity of subtraction of divergences for the operator in the r.h.s. of (78) and then this breaking term is definitively compensated by:

$$\hat{\Gamma}_{Eff} \rightarrow \hat{\Gamma}_{Eff} - \int d^4x \Omega_\mu^a \Lambda_{\mu\nu}^{ab} \hat{\gamma}_b^\nu \quad (81)$$

recovering the equation:

$$\frac{\delta \hat{\Gamma}}{\delta \Omega_\mu^a} = 0 \quad (82)$$

at every order of the perturbative expansion. This equation is also satisfied by the functional $\bar{\Gamma}$, because the necessary counterterms to compensate the Ω dependent breaking terms are independent from Ω itself. This is really important because it leads to the end of our demonstration.

Let us take the derivative of both sides of equation (78) with respect to Ω_μ^a and then put it to zero:

$$\frac{\delta}{\delta \Omega_\mu^a} (\mathcal{D}_{\bar{\Gamma}} \bar{\Gamma})|_{\Omega=0} = \frac{\delta}{\delta \Omega_\mu^a} (\tilde{\Delta}_{ST}^{(n)})|_{\Omega=0} \quad (83)$$

The r.h.s. is vanishing because of (77) and the first one can be written in the form:

$$\frac{\delta}{\delta \Omega_\mu^a} (\mathcal{D}_{\bar{\Gamma}} \bar{\Gamma})|_{\Omega=0} = \frac{\delta}{\delta \Omega_\mu^a} (\mathbf{D}_{\bar{\Gamma}} \bar{\Gamma} + \mathcal{R} \bar{\Gamma})|_{\Omega=0} \quad (84)$$

that is:

$$\frac{\delta}{\delta \Omega_\mu^a} (\mathcal{D}_{\bar{\Gamma}} \bar{\Gamma})|_{\Omega=0} = \frac{\delta}{\delta \Omega_\mu^a} (\mathbf{D}_{\bar{\Gamma}} \bar{\Gamma})|_{\Omega=0} + \left[\frac{\delta \bar{\Gamma}}{\delta V_\mu^a} \right]|_{\Omega=0} = 0 \quad (85)$$

Because of the eq. (82) and observing that the operator $\mathbf{D}_{\bar{\Gamma}}$ commutes with the functional derivative $\frac{\delta}{\delta \Omega_\mu^a}$, we have

$$\left(\frac{\delta \bar{\Gamma}}{\delta V_\mu^a} \right) = 0 \quad (86)$$

for every value of the field Ω_μ^a and for every order of perturbation expansion. This allows us to deduce that the ST Identities are reduced to the following:

$$\mathbf{D}_{\bar{\Gamma}} \bar{\Gamma} = \hbar^n \tilde{\Delta}_{ST}^{(n)} + O(\hbar^{n+1}) \quad (87)$$

for the background field independent functional $\bar{\Gamma}$, i.e. we recover the conventional ST identities:

$$\int d^4x \left[\frac{\delta \bar{\Gamma}}{\delta A_\mu^a} \frac{\delta \bar{\Gamma}}{\delta \hat{\gamma}_a^\mu} + \frac{\delta \bar{\Gamma}}{\delta \omega^a} \frac{\delta \bar{\Gamma}}{\delta \zeta_a} \right] = \hbar^n \tilde{\Delta}_{ST}^{(n)} + O(\hbar^{n+1}) \quad (88)$$

We remand to the literature for the study of breaking terms of eq. (88). We have now to confirm our hypotheses for the breakings of the equation (73).

5.3 Solution of consistency equation

The open problem is to find a solution of the consistency equation for the breakings (72)

$$\int d^4x \Omega_\mu^a(x) \frac{\delta}{\delta V_\mu^a} \int d^4y \Omega_\mu^a \tilde{P}_{1a}^{(n)\mu}(y) = 0 \quad (89)$$

in the space of local functional $(\Omega_\mu^a \tilde{P}_{1a}^{(n)\mu})(y)$ linear in the external field $\Omega_\mu^a(x)$, polynomial into the field $A_\mu^a(x)$ e $V_\mu^a(x)$ and in their derivative. On this space we shall find that the solution can be put in the form:

$$\int d^4y \Omega_\mu^a \tilde{P}_{1a}^{(n)\mu} = \mathcal{R} \sum_i \hat{\Gamma}_i \quad (90)$$

It belongs to the image of \mathcal{R} on integrated functionals $\hat{\Gamma}_i(A, V) = \int d^4x \Gamma_i(x)$ neutral under $Q_{\Phi\Pi}$ charge and with dimension 4.

We rewrite the \mathcal{R} operator in a more synthetic way :

$$\mathcal{R} = \int d^4x \Omega^i(x) \frac{\delta}{\delta \phi^i(x)} \quad (91)$$

and notice that, by the commutation properties of fields $\{\Omega^i(x)\}$ e $\{\phi^i(x)\}$, \mathcal{R} is nilpotent.

Changing a little the notation we want to give some account of the structure of the linear space \mathcal{S} of local functional $\Gamma(x)$ [16], polynomial in Ω^i , ϕ^i and their derivatives² $D_{\mu(n)}\Omega^i(x), D_{\mu(n)}\phi^i(x)$.

On \mathcal{S} \mathcal{R} reduces to the local operator

$$\mathcal{R} = \sum_n D_{\mu(n)}\Omega^i(x) \frac{\partial}{\partial D_{\mu(n)}\phi^i(x)} \quad (92)$$

Since \mathcal{R} is nilpotent the image $Im(\mathcal{R})$ is a subspace of its kernel

$$Im(\mathcal{R}) \subseteq \ker(\mathcal{R}) \quad (93)$$

then our problem is to find out the possible solutions $\hat{\Gamma}(x)$ of

$$\mathcal{R}\Gamma(x) = 0 \quad (94)$$

that do not belong to $Im(\mathcal{R})$ or, in other terms, the representatives of the cohomology classes $H(\mathcal{R})$:

$$H(\mathcal{R}) \equiv \frac{\ker(\mathcal{R})}{Im(\mathcal{R})} \quad (95)$$

Our space of functionals \mathcal{S} is an Hilbert space³, so that it is possible to define the adjoint of \mathcal{R} as:

$$\mathcal{R}^\dagger = \sum_n D_{\mu(n)}\phi^i(x) \frac{\partial}{\partial D_{\mu(n)}\Omega^i(x)} \quad (96)$$

² In this section we will use the notation $D_{\mu(n)}$ for the collective derivative $D_{\mu(n)}\Omega^i(x) \equiv \partial_{\mu_1} \dots \partial_{\mu_n}\Omega^i(x)$ and $D^{(0)}\Omega^i(x) \equiv \Omega^i(x)$. By the locality of the elements of the space \mathcal{S} these variables are functional independent.

³More accounts in the appendix B

and the correspondent selfadjoint Laplace-Beltrami operator:

$$\Delta \equiv \{\mathcal{R}, \mathcal{R}^\dagger\} = \sum_n \left(D_{\mu(n)} \Omega^i(x) \frac{\partial}{\partial D_{\mu(n)} \Omega^i(x)} + D_{\mu(n)} \phi^i(x) \frac{\partial}{\partial D_{\mu(n)} \phi^i(x)} \right) \quad (97)$$

where kernel ($\ker \Delta$) is the subspace of harmonic functionals:

$$\Delta \Gamma(x) = 0 \quad (98)$$

Thanks to the Hilbert structure of \mathcal{S} for every cohomology class $[\Gamma(x)] \in H(\mathcal{R})$ there is only one harmonic representative. Then the cohomolgy class of the operator \mathcal{R} and of its adjoint \mathcal{R}^\dagger is identified with the space $\ker \Delta$, or more specifically we have: $\ker(\mathcal{R}) \cap \ker(\mathcal{R}^\dagger) = \ker \Delta$.

Form the expression of the Δ we find that the space $\ker \Delta$ is nothing else but the space $\hat{\mathcal{S}}$ of local functionals independent on field variables $D_{\mu(n)} \Omega^i(x)$ and $D_{\mu(n)} \phi^i(x)$. Thus every solution $\Gamma(x)$ of (94) is given by:

$$\Gamma(x) = \mathcal{R} \Gamma'(x) + \hat{\Gamma}(x) \quad \Gamma'(x) \in \mathcal{S} \quad \hat{\Gamma}(x) \in \hat{\mathcal{S}} \quad (99)$$

We now may come back to the space of integrated functionals for which the consistency equation is

$$\mathcal{R} \int d^4 x \Gamma(x) = 0 \quad (100)$$

We recall that the integration is extended to whole Euclidean space and the conventional hypothesis of smoothness for the functionals of \mathcal{S} are assumed. Besides the Hilbert space structure we introduce the p-forms $\omega_p^q(x)$ with $Q_{\Phi\Pi} = q$ and contract the Lorentz indices with 1-forms dx^μ . In particular we are interested in the solution of (100) on the 4-forms $\omega_4^0(x)$ subspace.

The equation admits two types of solutions: the cocycles and the d-module cocycles

$$\mathcal{R} \omega_4^0(x) = d \omega_3^1(x) \quad (101)$$

where d is the conventional external derivative on p-forms spaces. Since the nilpotent operator \mathcal{R} commutes with d, acting by the left on both sides of (101) we obtain:

$$(\mathcal{R})^2 \omega_4^0(x) = 0 = d(\mathcal{R} \omega_3^1(x)) \quad (102)$$

which implies that the 3-form $\omega_3^1(x)$ is a d-cocycles. By the Poincaré theorem we find:

$$\mathcal{R} \omega_3^1(x) = d \omega_2^2(x) \quad (103)$$

i.e. the $\mathcal{R} \omega_3^1(x)$ form is a d-coboundary. Iterating the process we find the following ladder of descendent equations:

$$\begin{aligned} \mathcal{R} \omega_4^0(x) &= d \omega_3^1(x) \\ \mathcal{R} \omega_3^1(x) &= d \omega_2^2(x) \\ \mathcal{R} \omega_2^2(x) &= d \omega_1^3(x) \\ \mathcal{R} \omega_1^3(x) &= d \omega_0^4(x) \\ \mathcal{R} \omega_0^4(x) &= 0 \end{aligned} \quad (104)$$

By the previous result (99), the last equation of (104) gives:

$$\omega_0^4(x) = \mathcal{R}\chi_0^3(x) + c_0(x) \quad (105)$$

The 0-form c_0 is independent from variables $D_{\mu(n)}\Omega^i(x)$, $D_{\mu(n)}\phi^i(x)$ and, since we search for local solutions, it is a constant. Acting with external derivative on (105) we have:

$$d\omega_0^4(x) = d\mathcal{R}\chi_0^3(x) + dc_0 = \mathcal{R}(d\chi_0^3) \quad (106)$$

and then, from (104), we arrive at:

$$\mathcal{R}(\omega_1^3 - d\chi_0^3) = 0. \quad (107)$$

Again, by (99), we can iterate the result:

$$\begin{aligned} \omega_1^3 &= d\chi_0^3 + \mathcal{R}\chi_1^2 + c_1 \\ \omega_2^2 &= d\chi_1^2 + \mathcal{R}\chi_2^1 + c_2 \\ \omega_3^1 &= d\chi_2^1 + \mathcal{R}\chi_3^0 + c_3 \\ \omega_4^0 &= d\chi_3^0 + \mathcal{R}\chi_4^{-1} + c_4 \end{aligned} \quad (108)$$

where the forms c_i have constant coefficients and then they can be written as $c_i = d\hat{c}_{i-1}$. Therefore, from that sequence, we obtain:

$$\omega_4^0 = \mathcal{R}\chi_4^{-1} + c_4 + d\chi_3^0. \quad (109)$$

The integrated functional

$$\int \omega_4^0 = \mathcal{R} \int \chi_4^{-1} + \int c_4 \quad (110)$$

is the solution of consistency equation (100).

The second term on the r.h.s. of the previous equation belongs to the cohomology class of \mathcal{R} , which we identified with the class of harmonic functionals independent on fields $D_{\mu(n)}\Omega^i(x)$, $D_{\mu(n)}\phi^i(x)$. Since we are interested in the solution in the space of forms ω_4^0 linear dependent on Ω_μ^a , the term c_4 necessarily vanishes. The solution of (110) allows us to express the $\Omega_\mu^a(x)$ -dependent breakings of ST identities, which solve the consistency equation (89), as R-variations of integrated functionals. The analysis of the previous section shows how employ this result to cancel up to the \hbar^n -order every Ω_μ^a -dependent breaking of $\Delta_{ST}^{(n)}$. Notice also that we have not used the WT identities of background gauge invariance. The next section is devoted to the study of these important identities.

5.4 Renormalization of Ward-Takahashi Identities

We have left this set of equations as the last to be renormalized because, as we will see, we will derive benefit from the conclusions of the previous sections. The necessary compensation of breakings of ST identities does not spoil further the WT identities since we considered these latter as broken by all possible sorts of breaking terms. Hence we can

replace the 1PI functional by the modified one $\bar{\Gamma}$, which satisfies eq. (86), in our WT identities:

$$W(\lambda)\bar{\Gamma} = \hbar^n \Delta_W^{(n)}(\lambda) + O(\hbar^{n+1}) \quad (111)$$

Again we suppose the renormalizability up to the \hbar^n -order.

We have to recall the consistency equations to which the breakings $\Delta_W^{(n)}(\lambda)$ must satisfy:

$$W(\lambda)\Delta_{ST}^{(n)} - \mathcal{F}_{\hat{\Gamma}_0}\Delta_W^{(n)}(\lambda) = 0 \quad (112)$$

and

$$W(\lambda)\Delta_W^{(n)}(\beta) - W(\beta)\Delta_W^{(n)}(\lambda) = \Delta_W^{(n)}(\lambda \wedge \beta). \quad (113)$$

In the first one we may also replace the ST breakings $\Delta_{ST}^{(n)}$ with the Ω -independent terms $\tilde{\Delta}_{ST}^{(n)}$:

$$W(\lambda)\tilde{\Delta}_{ST}^{(n)} - \mathcal{F}_{\hat{\Gamma}_0}\Delta_W^{(n)}(\lambda) = 0 \quad (114)$$

It is easy to show the following commutation properties of $W(\lambda)$ and $\mathcal{F}_{\hat{\Gamma}_0}$ operators with the functional derivative with respect the Ω field:

$$\left[W(\lambda), \frac{\delta}{\delta\Omega_\mu^a} \right] = \left(\lambda \wedge \frac{\delta}{\delta\Omega} \right)_\mu^a \quad \left\{ \mathcal{F}_{\hat{\Gamma}_0}, \frac{\delta}{\delta\Omega_\mu^a} \right\} = \frac{\delta}{\delta V_\mu^a} \quad (115)$$

by which we obtain:

$$\left[W(\lambda), \frac{\delta}{\delta\Omega_\mu^a} \right] \tilde{\Delta}_{ST}^{(n)} - W(\lambda) \left(\frac{\delta}{\delta\Omega_\mu^a} \tilde{\Delta}_{ST}^{(n)} \right) + \mathcal{F}_{\hat{\Gamma}_0} \left(\frac{\delta}{\delta\Omega_\mu^a} \Delta_W^{(n)}(\lambda) \right) - \frac{\delta}{\delta V_\mu^a} \Delta_W^{(n)}(\lambda) = 0 \quad (116)$$

Due to the results of the previous sections that our breaking terms $\tilde{\Delta}_{ST}^{(n)}$ are independent from Ω we have that the two terms of l.h.s. of equation (116) vanish and then we can replace the eq. (116) with the easier one:

$$\mathcal{F}_{\hat{\Gamma}_0} \left(\frac{\delta}{\delta\Omega_\mu^a} \Delta_W^{(n)}(\lambda) \right) - \frac{\delta}{\delta V_\mu^a} \Delta_W^{(n)}(\lambda) = 0 \quad (117)$$

If we success to prove in another way that the breakings $\Delta_W^{(n)}(\lambda)$ are Ω -independent we may conclude:

$$\left(\frac{\delta}{\delta V_\mu^a} \right) \Delta_W^{(n)}(\lambda) = 0. \quad (118)$$

We have now to make explicit the structure of $\Delta_W^{(n)}(\lambda)$. In the first we split the external field $(\hat{\gamma}, \zeta, \Omega)$ part from the independent one:

$$\Delta_W^{(n)}(\lambda) = \int d^4x \left\{ a_1 tr[\lambda \hat{\gamma}_\mu \Delta_\mu^\lambda] + a_2 tr[\lambda \Omega_\mu \Delta_\mu^\Omega] + a_3 tr[\lambda \zeta \Delta^\zeta] + a_4 tr[\lambda \Delta(A, V)] \right\} \quad (119)$$

Here the trace is taken over matrices of adjoint representation so that every term is invariant under global gauge transformations. The field λ is a test function carrying an

index in the same representation of fields. The local polynomials Δ_μ^Ω , $\Delta(A, V)$, $\Delta_\mu^{\hat{\gamma}}$ and Δ^ζ are of increasing powers of $Q_{\Phi\Pi}$ charge from -1 and all of dimension 2 except of external field independent term $\Delta(A, V)$ of dimension 4.

One immediately sees that the breakings Δ_μ^Ω and Δ^ζ are of the form:

$$(\Delta^\zeta)^{ab} = \frac{1}{2}\Lambda^{ab[cd]}\omega_c\omega_d \quad (\Delta_\mu^\Omega)^{ab} = \Lambda^{abc}\hat{\gamma}^c \quad (120)$$

but these terms do not satisfy the Wess-Zumino consistency equation (113) and then $a_3 = a_2 = 0$. For the $\Delta_\mu^{\hat{\gamma}}$ one finds the possible monomial decomposition:

$$tr[\lambda\hat{\gamma}_\mu\Delta_\mu^{\hat{\gamma}}] = \Lambda_1^{abc}\lambda^a\partial^\mu(\hat{\gamma}_\mu^b\omega^c) + \Lambda_2^{abc}\lambda^a\hat{\gamma}_\mu^b\partial^\mu\omega^c + \Lambda_3^{abcd}\lambda^a\hat{\gamma}_\mu^bV_\mu^c\omega^d + \Lambda_4^{abcd}\lambda^a\hat{\gamma}_\mu^bA_\mu^c\omega^d \quad (121)$$

It is direct to show that only the first term satisfy the Wess-Zumino equation ((113)) then we have to seek, if exists, its counter term, but we have:

$$W(\beta)\int d^4x tr[\omega\hat{\gamma}_\mu V_\mu] = -\int d^4x tr[\beta\partial_\mu(\hat{\gamma}_\mu\omega)] \quad (122)$$

Thus introducing this counter term into the effective action up to the \hbar^n -order, we have only to consider the external field independent terms $\Delta(A, V)$. One has to note also that this counter term spoils again the ST identities, but, by the theorem proved before, we have always the possibility to adjust finely the coefficients of counter terms in order to cancel the Ω_μ^a -dependent breakings both for ST and for WT identities. By the previous result ((118)), we have that $\Delta(A, V)$ is independent from the background field V_μ^a and then the only possible anomalies is the conventional ABJ ([1]).

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Appendix

In this appendix we want to describe shortly the Hilbert structure of the \mathcal{S} of local polynomial functionals used in the previous sections, then we want to give just a little account about the Hodge decomposition theorem on this space.

We have been forced to introduce the base of the space \mathcal{S} defined by independent variables $D_{\mu(n)}\Omega^i(x)$ and $D_{\mu(n)}\phi^i$ with opposite commutative character, on the same ground we introduce the adjoint base: $D_{\mu(n)}\Omega^{i\dagger}(x)$, $D_{\mu(n)}\phi^{i\dagger}$ with the relations:

$$\begin{aligned} \{D_{\mu(m)}\Omega^i(x), D_{\nu(n)}\Omega^{j\dagger}(x)\} &= \delta_{m,n}\delta_{ij}\delta_{\nu(1)\dots\nu(n)}^{\mu(1)\dots\mu(m)} \\ [D_{\mu(m)}\phi^i(x), D_{\nu(n)}\phi^{j\dagger}(x)] &= \delta_{m,n}\delta_{ij}\delta_{\nu(1)\dots\nu(n)}^{\mu(1)\dots\mu(m)} \end{aligned} \quad (123)$$

where $\delta_{\nu(1)\dots\nu(n)}^{\mu(1)\dots\mu(m)}$ is a notation for symmetrised Kronecker symbol. From this definition it is easy to show that \mathcal{R} is the adjoint operator, and to show that Laplace-Beltrami operator

$$\Delta \equiv \mathcal{R}\mathcal{R}^\dagger + \mathcal{R}^\dagger\mathcal{R} \quad (124)$$

is selfadjoint.

Decomposition theorem. The vector space \mathcal{S} could be spitted into direct sum of orthogonal subpace:

$$\mathcal{S} = Im(\mathcal{R}) \oplus Im(\mathcal{R}^\dagger) \oplus (\ker(\mathcal{R}) \cap \ker(\mathcal{R}^\dagger)) \quad (125)$$

Proof. From the decomposition:

$$\mathcal{S} = Im\Delta \oplus \ker \Delta \quad (126)$$

every vector ω can be written in the form:

$$\omega = \Delta\chi + \xi = \{\mathcal{R}\mathcal{R}^\dagger + \mathcal{R}^\dagger\mathcal{R}\}\chi + \xi \quad (127)$$

where $\Delta\xi = 0$, and such as, if we set $\chi' = \mathcal{R}^\dagger\chi$, $\chi'' = \mathcal{R}\chi$, we have

$$\omega = \Delta\chi + \xi = \mathcal{R}\chi' + \mathcal{R}^\dagger\chi'' + \xi \quad (128)$$

and, by $0 = (\xi, \Delta\xi) = (\mathcal{R}\xi, \mathcal{R}\xi) + (\mathcal{R}^\dagger\xi, \mathcal{R}^\dagger\xi)$ we have immediately:

$$\mathcal{R}\xi = \mathcal{R}^\dagger\xi = 0 \quad (129)$$

i.e. our thesis.

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